JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 28, No. 1, January–February 2005

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Differentials, Variations, and Taylor Series

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Introduction

THE purpose of this Note is to establish the relationship between making a Taylor series expansion and taking a differential for problems involving ordinary differential equations and algebraic boundary conditions. Once this has been done, a simple perturbation problem is used to demonstrate the mechanics of taking differentials. Then a standard optimal control problem is used to show that the use of differentials makes it possible to unify the various theories for finding a relative minimum (parameter optimization, calculus of variations, and optimal control theory).

The relationship between Taylor series and differentials is known for algebraic equations.[1] However, the process of taking differentials has not been included in recent engineering mathematics texts.[2]

Two Notes previously published on this subject[3,4] used the terminology of a time-fixed variation and a time-free variation. The latter is actually the differential of calculus, and the former is the variation of the calculus of variations.

It was stated in Ref. 5 (p. 6) that there appeared to be analogies between the differentials of ordinary calculus and variations of the calculus of variations. It was also stated that, as of 1946, the issue had not been resolved. In Ref. 6 (1975), it appears that some relationship was established, but it is not clear or consistent. For example, on p. 102 the first-order terms are called the first variation and are supposedly obtained by taking a variation. However, the expression for the first variation contains differentials that could only have been obtained if the differential had been taken. Throughout Ref. 6 it is not clear if the first- and second-order terms are obtained by taking variations (differentials) or are the results from Taylor series expansions. Reference 7 is an improvement, but it is not completely consistent. The uniform application of differentials has been one motivation for writing Ref. 8.

In this Note, the process of taking differentials is reviewed for algebraic equations. Then the process is discussed for ordinary differential equations and is applied to a perturbation problem and an optimal control problem.

Presented as Paper 2003-0666 at the AAS/AIAA Astrodynamics Specialist Conference, Big Sky, MT, 3–7 August 2003; received 11 December 2003; revision received 13 May 2004; accepted for publication 13 May 2004.

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Taylor Series and Differentials

Consider the equation

$$y = y(x) \tag{1}$$

where x is the independent variable and y is the dependent variable. Consider two points on curve (1): a nominal point x and a neighboring point $x_* = x + \mathrm{d}x$, where $\mathrm{d}x$ is small displacement, assumed constant. It is desired to express $y(x+\mathrm{d}x)$ in terms of y and its derivatives at x. The Taylor series expansion of $y(x+\mathrm{d}x)$ is given by

$$y(x+\mathrm{d}x) = y(x) + y_x(x)\,\mathrm{d}x + \tfrac{1}{2!}y_{xx}(x)\,\mathrm{d}x^2 + \cdots \tag{2}$$

where the subscript x denotes a derivative with respect to the general x and the argument x means evaluated at the nominal x.

The differential of the function $y(x)$ is given by $\mathrm{d}y = y_x(x)\,\mathrm{d}x$. The second differential becomes $\mathrm{d}^2y = y_{xx}\,\mathrm{d}x^2$ because $\mathrm{d}x$ is constant; that is, $\mathrm{d}(\mathrm{d}x) = 0$. Hence, each order term of Eq. (2) excluding the numerical coefficient is the same-order differential of $y(x)$ evaluated at the nominal point. The total change in y, that is, $\Delta y \triangleq y(x+\mathrm{d}x) - y(x)$, can be rewritten as

$$\Delta y = \mathrm{d}y + \tfrac{1}{2!}\mathrm{d}^2y + \cdots \tag{3}$$

where

$$\mathrm{d}y = y_x(x)\,\mathrm{d}x, \qquad \mathrm{d}^2y = y_{xx}(x)\,\mathrm{d}x^2, \ldots \tag{4}$$

The total change in the dependent variable y has a first-order part in $\mathrm{d}x$, a second-order part in $\mathrm{d}x$, and so on. The total change in the independent variable only has a first-order part because $\mathrm{d}x$ is constant ($\Delta x = \mathrm{d}x$).

The conclusion is that the Taylor series can be developed one term at a time by taking differentials, with the understanding that the differential of an independent variable is constant (Ref. 7, p. 81).

Differentials: Algebraic Equations

Consider the scalar algebraic equation

$$\psi(x, y) = 0 \tag{5}$$

where x is the independent variable and y is the dependent variable. Equation (5) is the implicit form of Eq. (1) and is the form of a boundary condition to be encountered later. On the curve defined by Eq. (5), consider two neighboring points: the nominal point x, y and the perturbed point $x_* = x + \mathrm{d}x$, $y_* = y + \Delta y$. It is desired to express the perturbed point in terms of ψ and its derivatives at the nominal point. For the perturbed point, $\mathrm{d}x$ is constant, and $\Delta y = \mathrm{d}y + \mathrm{d}^2y/2! + \cdots$. Next, $\psi(x_*, y_*)$ is expanded in terms of $\mathrm{d}x$ and Δy, which is then written in terms of differentials. This process leads to

$$\psi(x_*, y_*) = \psi(x, y) + \mathrm{d}\psi(x, y) + \tfrac{1}{2!}\mathrm{d}^2\psi(x, y) + \cdots = 0 \tag{6}$$

By neglecting higher order terms, it is seen that $\mathrm{d}\psi(x, y) = 0$. Similarly, $\mathrm{d}^2\psi(x, y) = 0$, and so on. Hence, the first-order part of Eq. (6) satisfies the relation

$$\mathrm{d}\psi(x, y) \triangleq \psi_x(x, y)\,\mathrm{d}x + \psi_y(x, y)\,\mathrm{d}y = 0 \tag{7}$$

and the second-order part is given by

$$d^2\psi(x, y) \triangleq \psi_{xx}(x, y)\,dx^2 + 2\psi_{xy}(x, y)\,dx\,dy + \psi_{yy}(x, y)\,dy^2 + \psi_y(x, y)\,d^2y = 0 \tag{8}$$

and so on. Given x, y, and dx, these equations can be solved for dy, d^2y, and so on in terms of dx to obtain Δy.

An example of an algebraic perturbation problem is the solution of Kepler's equation $E - e\sin E - M = 0$ for $E = E(e, M)$ for small eccentricity e. This problem is discussed in Ref. 3, where the variation δ should be replaced by the differential d.

An example of an algebraic optimization problem is the minimization of the scalar performance index $J = \phi(x)$ subject to the equality constraint $\psi(x) = 0$. There are n variables and $p < n$ constraints. After the augmented performance index $J' = \phi(x) + \nu^T\psi(x)$ is formed (ν is a constant Lagrange multiplier), the conditions for a relative minimum are the following: the first differential must vanish for an optimal point ($dJ' = 0$) and, if the second differential is positive ($d^2J' > 0$), the optimum is a minimum.

Differentials: Differential Equations

Consider two neighboring paths (Fig. 1): the nominal path $x(t)$ and the neighboring or perturbed path $x_*(t)$, which at this point is assumed to be caused by a perturbation in the initial conditions. Each path is the solution of the differential equation $\dot{x} = f(t, x)$. Two paths are said to be neighboring if, at time t, $x_*(t)$ is close to $x(t)$, $\dot{x}_*(t)$ is close to $\dot{x}(t)$, and so on. This means that

$$x_* = x + \tilde{\Delta}x, \qquad \dot{x}_* = \dot{x} + \tilde{\Delta}\dot{x} \tag{9}$$

and so on, where $\tilde{\Delta}$ is the total change at constant time. While $\tilde{\Delta}x$ is a small quantity, it has a part δx that is proportional to the first power of the perturbation that causes $\tilde{\Delta}x$, a part $\delta^2 x$ that is proportional to the second power of the perturbation, and so on. The same can be said about $\tilde{\Delta}\dot{x}$. Hence, $\tilde{\Delta}x$ and $\tilde{\Delta}\dot{x}$ can be written as

$$\tilde{\Delta}x = \delta x + (1/2!)\delta^2 x + \cdots, \qquad \tilde{\Delta}\dot{x} = \delta\dot{x} + (1/2!)\delta^2\dot{x} + \cdots \tag{10}$$

and so on.

Neighboring points lie on neighboring paths and are in the neighborhood of each other. Figure 1 shows neighboring points at the same time t and neighboring points at different times t_* and t.

A well-known relation can be derived at this point. Note from Eqs. (9) that

$$\frac{d}{dt}\tilde{\Delta}x = \tilde{\Delta}\dot{x} \tag{11}$$

With Eqs. (10), Eq. (11) becomes, to first order,

$$\frac{d}{dt}\delta x = \delta\frac{dx}{dt} \tag{12}$$

which means that the derivative d/dt and the symbol δ interchange.

For a perturbed point that is at a different but neighboring time,

$$\Delta x = x_*(t_*) - x(t) \tag{13}$$

$$\Delta t = t_* - t \tag{14}$$

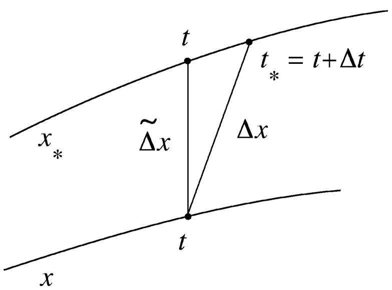

Fig. 1 Neighboring paths and neighboring points.

The changes Δx and Δt are small, but they have first-order parts, second-order parts, and so on, and are expressed as

$$\Delta x = dx + (1/2!)\,d^2x + \cdots \tag{15}$$

$$\Delta t = dt + (1/2!)\,d^2t + \cdots \tag{16}$$

The time-free change Δx is related to the time-fixed change $\tilde{\Delta}x$ and Δt as follows:

$$\begin{aligned}\Delta x &= x_*(t_*) - x(t)\\ &= x_*(t_*) - x_*(t) + x_*(t) - x(t)\\ &= \tilde{\Delta}x + x_*(t + \Delta t) - x_*(t)\\ &= \tilde{\Delta}x + \dot{x}_*(t)\Delta t + (1/2!)\ddot{x}_*(t)\Delta t^2 + \cdots\\ &= \tilde{\Delta}x + (\dot{x} + \tilde{\Delta}\dot{x})\Delta t + \cdots\\ \Delta x &= \tilde{\Delta}x + \dot{x}\Delta t + \cdots\end{aligned} \tag{17}$$

where $\dot{x}$ is evaluated on the nominal path. Then, with Eqs. (10), (15), and (16), the first-order part of the last of Eqs. (17) is given by the well-known formula

$$dx = \delta x + \dot{x}\,dt \tag{18}$$

This result defines the relationship between dx and δx, dt.

Although Eq. (18) is derived for $x(t)$, it could have been derived for an arbitrary function of time, say $A(t)$; that is,

$$dA(t) = \delta A(t) + [dA(t)/dt]\,dt \tag{19}$$

Now consider the scalar differential equation

$$\dot{x} = f(t, x) \tag{20}$$

It is desired to express the neighboring path in terms of f and its derivatives evaluated on the nominal path. Along the neighboring path,

$$\dot{x}_* = f(t, x_*) \tag{21}$$

where x_* is assumed to be caused by a small change in the initial conditions. Since $x_* = x + \tilde{\Delta}x$, a Taylor series expansion leads to

$$0 = \frac{d}{dt}\tilde{\Delta}x - f_x(t, x)\tilde{\Delta}x + \cdots = \frac{d}{dt}\delta x - f_x(t, x)\delta x + \cdots \tag{22}$$

Because of Eq. (12), it can be rewritten as

$$0 = \delta\dot{x} - f_x(t, x)\delta x + \cdots \tag{23}$$

and indicates that the first-order term is the differential of $\dot{x} - f(t, x)$ holding the time constant, evaluated on the nominal path. To connect with the calculus of variations, the time-constant differential is the variation. Taking the variation of Eq. (20) shows that the first-order term satisfies the equation

$$\delta\dot{x} = f_x(t, x)\delta x \tag{24}$$

so that the differential equation for δx is given by

$$\frac{d}{dt}\delta x = f_x(t, x)\delta x \tag{25}$$

In summary, the differential equation for the first-order part of the total change is given by the variation of differential equation (20) evaluated on the nominal path. The second-order part comes from the second variation of Eq. (20) evaluated on the nominal path, and so on.

Perturbation Problem

In Ref. 4, two perturbation problems are considered. The first is orbital motion in the equatorial plane of an oblate spheroid Earth. The small parameter is the flatness of the Earth. The second is the derivation of the Clohessy–Wiltshire equations, where the perturbed path is caused by a small perturbation in the initial state. Both of these problems involve multiple differential equations, but the final condition is just the given final time. To correct the terminology with respect to differentials and variations, a simple perturbation problem is examined here.

An example of a perturbation problem is that of finding the function $x(t)$ that satisfies the scalar ordinary differential equation and boundary conditions

$$\dot{x} = f(t, x, \varepsilon), \qquad t_0 = t_{0_s}, \qquad x_0 = x_{0_s}, \qquad \psi(t_f, x_f) = 0 \tag{26}$$

where ε is a small parameter (Fig. 2), $f = \alpha(t, x) + \varepsilon\beta(t, x)$, and the subscript s denotes a specific value. It is assumed that Eq. (26) cannot be solved analytically. However, if the zeroth-order equation ($\varepsilon = 0$)

$$\dot{x} = f(t, x, 0) \tag{27}$$

has an analytical solution, it is possible to obtain an improved analytical solution for $\varepsilon \neq 0$. Note that the symbol x denotes the general solution of Eq. (26), and it also denotes the solution of Eq. (27) where $\varepsilon = 0$.

The first-order part of the perturbed path is obtained from the variation of the differential equation, and the first-order part of the boundary conditions (algebraic equations) is obtained from the differential of the boundary conditions. After interchanging δ and $\mathrm{d}/\mathrm{d}t$, they are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x = f_x(t, x, \varepsilon)\delta x + f_\varepsilon(t, x, \varepsilon)\delta\varepsilon$$

$$\mathrm{d}t_0 = 0, \qquad \mathrm{d}x_0 = 0, \qquad \psi_{t_f}(t_f, x_f)\,\mathrm{d}t_f + \psi_{x_f}(t_f, x_f)\,\mathrm{d}x_f = 0 \tag{28}$$

and, evaluated on the nominal path ($\varepsilon = 0$), become

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta x = f_x(t, x, 0)\delta x + f_\varepsilon(t, x, 0)\delta\varepsilon$$

$$t_0 = 0, \qquad \delta x_0 = 0, \qquad \psi_{t_f}(t_f, x_f)\,\mathrm{d}t_f + \psi_{x_f}(t_f, x_f)\,\mathrm{d}x_f = 0 \tag{29}$$

Combined with the equation

$$\mathrm{d}x_f = \delta x_f + \dot{x}_f\,\mathrm{d}t_f \tag{30}$$

which is Eq. (16) evaluated at the final point, these equations can be solved for $\delta x(t)$, $\mathrm{d}t_f$, and $\mathrm{d}x_f$ for a given $\delta\varepsilon$. The values of $\mathrm{d}x$ for $0 < \mathrm{d}t < \mathrm{d}t_f$ can be obtained from the equation

$$\mathrm{d}x = \delta x_f + \dot{x}_f\,\mathrm{d}t \tag{31}$$

The equations for the second- and higher-order terms can be obtained from Eqs. (28) in a similar fashion with $\delta\varepsilon = \text{const}$; that is, $\delta(\delta\varepsilon) = 0$.

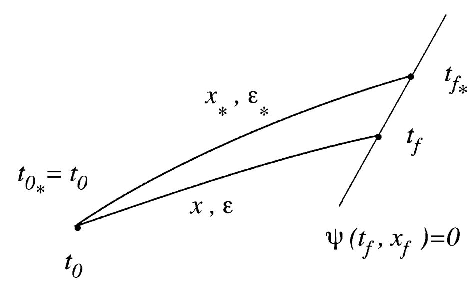

Fig. 2 Perturbation problem.

Optimal Control Problem

A well-known optimal control problem is the following: Find the control history $u(t)$ that minimizes the performance index

$$J = \phi(t_f, x_f) + \int_{t_0}^{t_f} L(t, x, u)\,\mathrm{d}t \tag{32}$$

subject to the differential constraints and boundary conditions

$$\dot{x} = f(t, x, u), \qquad t_0 = t_{0_s}, \qquad x_0 = x_{0_s}, \qquad \psi(t_f, x_f) = 0 \tag{33}$$

where the state x is an $n \times 1$ vector, and the control u is an $m \times 1$ vector. The quantities ϕ and L are scalars, and ψ is a $p \times 1$ vector where $0 < p \leq n + 1$; there must be at least one final condition that draws the optimal path to the final point. The subscript s denotes a specific value.

The first step in deriving the optimality conditions is to adjoin the constraints to the performance index by Lagrange multipliers ν and $\lambda(t)$ to form the augmented performance index:

$$J' = G(t_f, x_f, \nu) + \int_{t_0}^{t_f} [H(t, x, u, \lambda) - \lambda^T \dot{x}]\,\mathrm{d}t \tag{34}$$

The endpoint function G and the Hamiltonian H are defined as

$$G = \phi(t_f, x_f) + \nu^T \psi(t_f, x_f), \qquad H = L(t, x, u) + \lambda^T f(t, x, u) \tag{35}$$

Next, the value of J' along the minimal path (nominal path) is compared with that along a comparison path (perturbed path), J'_*, by forming the difference (Fig. 3)

$$\begin{aligned}\Delta J' &= J'_* - J' \\ &= G\big(t_{f_*}, x_{*_f}, \nu_*\big) + \int_{t_0}^{t_{f*}} \big[H(t, x_*, u_*, \lambda_*) - \lambda_*^T \dot{x}_*\big]\,\mathrm{d}t \\ &\quad - G(t_f, x_f, \nu) - \int_{t_0}^{t_f} [H(t, x, u, \lambda) - \lambda^T \dot{x}]\,\mathrm{d}t \end{aligned} \tag{36}$$

For u to be the control that makes J' a relative minimum, $\Delta J'$ must be positive regardless of the choice of the comparison path. The comparison path must satisfy all of constraints (33).

With reference to Fig. 3, the following definitions are introduced:

$$\begin{aligned} u_*(t) &= u(t) + \delta u(t) \\ x_*(t) &= x(t) + \delta x(t) + (1/2!)\delta^2 x(t) + \cdots \\ \lambda_*(t) &= \lambda(t) + \delta\lambda(t) + (1/2!)\delta^2\lambda(t) + \cdots \\ \nu_* &= \nu + \mathrm{d}\nu + (1/2!)\,\mathrm{d}^2\nu + \cdots \\ t_{f_*} &= t_f + \mathrm{d}t_f + (1/2!)\,\mathrm{d}^2 t_f + \cdots \\ x_*(t_{f_*}) &= x_f + \mathrm{d}x_f + (1/2!)\,\mathrm{d}^2 x_f + \cdots \end{aligned} \tag{37}$$

where δ denotes a variation and d is a differential. The comparison path is caused by δu, meaning that δu is fixed $[\delta(\delta u) = 0]$ and so it only has a first-order part. If these relations are substituted into

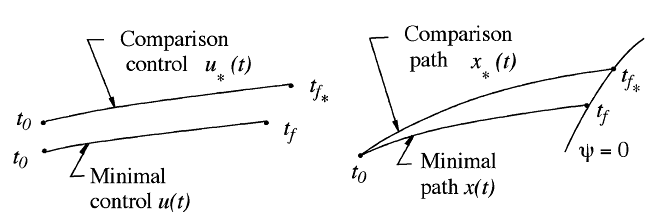

Fig. 3 Comparison control and path.

Eq. (36) and a Taylor series expansion is performed, the first-order part of $\Delta J'$ is given by

$$\Delta J' = G_{t_f}\,\mathrm{d}t_f + G_{x_f}\,\mathrm{d}x_f + G_\nu\,\mathrm{d}\nu + [(H - \lambda^T\dot{x})\,\mathrm{d}t]_{t_0}^{t_f} + \int_{t_0}^{t_f}\left(H_x\delta x + H_u\delta u + H_\lambda\delta\lambda - \delta\lambda^T\dot{x} - \lambda^T\delta\dot{x}\right)\mathrm{d}t + \cdots \tag{38}$$

The coefficients of the differentials and variations are evaluated on the minimal path, and so the coefficients of $\mathrm{d}\nu$ and $\delta\lambda$ are zero.

Now, the first three terms are the differential of G evaluated on the minimal path. The next two terms are the differential of the integral (Leibnitz's rule) evaluated on the minimal path. As a consequence,

$$\Delta J' = \mathrm{d}J' + (1/2!)\,\mathrm{d}^2J' + \cdots \tag{39}$$

where $\mathrm{d}J'$ is the differential of J' evaluated on the minimal path and d^2J' is the second differential of J' evaluated on the minimal path.

For a relative minimum, $\Delta J'$ must be positive regardless of the choice of the comparison path. Hence, the general conditions for a minimal control are that the first differential must vanish for an optimal control, that is,

$$\mathrm{d}J' = 0 \tag{40}$$

and if the second differential is positive, that is,

$$\mathrm{d}^2J' > 0 \tag{41}$$

the optimal control is a minimum.

Note that these are also the conditions for the parameter optimization problem. Because they can be extended to problems involving multiple integrals, the use of differentials unifies the various optimization theories.

In practice, it has become the convention to call Eqs. (40) and (41) the first and second variations, but the correct terminology is first and second differentials. However, the time-fixed differential is the variation of the calculus of variations.

Conclusions

The purpose of this Note has been to discuss the relationship between making a Taylor series expansion and taking differentials for problems involving ordinary differential equations and algebraic boundary conditions. Two such problems are the perturbation problem and the optimal control problem. In the perturbation problem, the differential equation involves a small parameter and/or a perturbation in the boundary conditions. In the optimal control problem, the differential equation has a control variable. In both problems, use is made of neighboring paths (nominal path and perturbed path) to derive the governing equations. In general, the perturbed path is expressed in terms of small changes from the nominal path, and the equations for the perturbed path are expanded in a Taylor series to obtain the first-order equations, the second-order equations, and so on. In this Note, it is shown that the Taylor series can be made by taking differentials of the problem equations and evaluating them on the nominal path. A perturbation problem and an optimal control problem have been used to demonstrate the process of taking differentials.

The use of differentials makes it possible to unify the various optimization theories: parameter optimization, optimal control theory, and calculus of variations. After the constraints are adjoined to the performance index with Lagrange multipliers to form the augmented performance index, the necessary conditions and the sufficient conditions for a relative minimum are obtained by applying the following general conditions: The first differential of the augmented performance index must vanish for an optimum and, if the second differential is positive, the optimum is a minimum. This is the case if the unknown is a point, a curve, a surface, and so on.

The relationship between the differential and the variation has also been established. The variation is a differential taken while holding the variable of integration (time here) constant.

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